

On α -embedded subsets of products

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Abstract

We prove that every continuous function $f : E \rightarrow Y$ depends on countably many coordinates, if E is an (\aleph_1, \aleph_0) -invariant pseudo- \aleph_1 -compact subspace of a product of topological spaces and Y is a space with a regular G_δ -diagonal. Using this fact for any $\alpha < \omega_1$ we construct an $(\alpha + 1)$ -embedded subspace of a completely regular space which is not α -embedded.

Keywords: κ -invariant set, pseudo- \aleph_1 -compact, α -embedded set
2000 MSC: Primary 54B10, 54C45; Secondary 54C20, 54H05

1. Introduction

If P is a property of functions, then by $P(X)$ ($P^*(X)$) we denote the collection of all real-valued (bounded) functions on a topological space X with the property P . By the symbol C we denote the property of continuity and let B_α be the property of being the function of the α -th Baire class, where $0 \leq \alpha < \omega_1$.

Recall that a subset A of a space X is *functionally closed (open) in X* , if there is $f \in C^*(X)$ with $A = f^{-1}(0)$ ($A = X \setminus f^{-1}(0)$).

The system of all functionally open (closed) subsets of a space X we denote by \mathcal{G}_0^* (\mathcal{F}_0^*). Assume that the classes \mathcal{G}_ξ^* and \mathcal{F}_ξ^* are defined for all $\xi < \alpha$, where $0 < \alpha < \omega_1$. Then, if α is odd, the class \mathcal{G}_α^* (\mathcal{F}_α^*) consists of all countable intersections (unions) of sets of lower classes, and, if α is even the class \mathcal{G}_α^* (\mathcal{F}_α^*) consists of all countable unions (intersections) of sets of lower classes. The classes \mathcal{F}_α^* for odd α and \mathcal{G}_α^* for even α are said to be *functionally additive*, and the classes \mathcal{F}_α^* for even α and \mathcal{G}_α^* for odd α are called *functionally multiplicative*. A set A is *functionally measurable*, if $A \in \bigcup_{0 \leq \alpha < \omega_1} (\mathcal{F}_\alpha^* \cup \mathcal{G}_\alpha^*)$. Notice that the σ -algebra of functionally measurable subsets of X is also called the σ -algebra of Baire sets.

*The paper was submitted on August, 23, 2013 to the Central European Journal of Mathematics, but in spite of positive referee's report the manuscript was withdrawn due to financial demand for publication and resubmitted to the European Journal of Mathematics

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An important role in the extension theory play z -embedded sets (a subset A of a topological space X is z -embedded in X , if for any functionally closed set F in A there exists a functionally closed set B in X such that $B \cap A = F$). In [8] for any $\alpha < \omega_1$ it was introduced the notion of an α -embedded set, i.e. such a set $A \subseteq X$ that every its subset B of the functionally multiplicative class α in A is the restriction on A of some set of the functionally multiplicative class α in X . Obviously, the class of 0-embedded sets coincides with the class of z -embedded sets. It is not hard to verify that any α -embedded set is β -embedded, if $\alpha \leq \beta$ [8, Proposition 2.5]. The converse proposition is not true as Theorem 2.6 from [8] shows. In that theorem there was constructed the example of the 1-embedded subset E of the product $X = [0, 1] \times \prod_{t \in [0,1]} X_t$, where $X_t = \mathbb{N}$

for all $t \in [0, 1]$, which is not 0-embedded in X . In the given paper we generalize the above-mentioned result from [8] and show that for any $\alpha < \omega_1$ there exists a set $E \subseteq X$, which is $(\alpha + 1)$ -embedded and is not α -embedded in X .

The convenient tool in the investigation of properties of an α -embedded subset E of a product $\prod_{t \in T} X_t$ is the fact that under some conditions on E every continuous function $f : E \rightarrow \mathbb{R}$ depends on countably many coordinates (see definitions in Section 2). S. Mazur in [10] introduced invariant sets under projection (see Definition 2.1(a)) and proved that every continuous function $f : E \rightarrow Y$ depends on countably many coordinates, if $E \subseteq \Sigma(a)$ for some $a \in E$ and E is invariant under projection, X_t is a metrizable separable space for each $t \in T$ and Y is a Hausdorff space with a G_δ -diagonal. R. Engelking [5] showed the same in the case, when E is a set which is invariant under composition (see Definition 2.1(b)) which is contained in $\Sigma(a)$ for some $a \in E$, X_t is a T_1 -space with countable base for each $t \in T$ and Y is a Hausdorff space in which every one-point set is G_δ (see also [7]). N. Noble and M. Ulmer [11] obtained the dependence on countably many coordinates of a continuous function $f : E \rightarrow Y$, if E is a subset of a pseudo- \aleph_1 -compact space $\prod_{t \in T} X_t$, which contains $\sigma(a)$ for some $a \in E$ and Y is a space with a regular G_δ -diagonal. The result of Noble and Ulmer was generalized by W.W. Comfort and I.S. Gotchev in [2]. Here we consider the so-called (\aleph_1, \aleph_0) -invariant subsets of products and, developing the methods of Mazur and of Noble and Ulmer, we show that every continuous function $f : E \rightarrow Y$ depends on countably many coordinates if E is an (\aleph_1, \aleph_0) -invariant pseudo- \aleph_1 -compact subspace of a product of topological spaces X_t and Y is a space with a regular G_δ -diagonal.

2. Some properties of pseudo- \aleph_1 -compact invariant sets

Let $(X_t : t \in T)$ be a family of non-empty topological spaces, $X = \prod_{t \in T} X_t$ and let $a = (a_t)_{t \in T}$ be a fixed point of X . For $S \subseteq T$ we denote by p_S the projection $p_S : X \rightarrow \prod_{t \in S} X_t$, where $p_S(x) = (x_t)_{t \in S}$ for each $x = (x_t)_{t \in T} \in X$; by x_S^a we denote the point with the coordinates $(y_t)_{t \in T}$, where $y_t = x_t$, if

$t \in S$ and $y_t = a_t$, if $t \in T \setminus S$. For a basic open set $U = \prod_{t \in T} U_t \subseteq X$ let $N(U) = \{t \in T : U_t \neq X_t\}$.

Definition 2.1. A set $E \subseteq X$ is called

- a) *invariant under projection* [10], if $x_S^a \in E$ for any $x \in E$ and $S \subseteq T$;
- b) *invariant under composition* [5], if for any $x, y \in E$ and $S \subseteq T$ we have $z = (z_t)_{t \in T} \in E$, where $z_t = x_t$ for every $t \in S$ and $z_t = y_t$ for every $t \in T \setminus S$.

Clearly, every invariant under composition set E is invariant under projection for any $a \in E$.

Following Engelking [5], M. Hušek in [7, p. 132] introduced a notion of a κ -invariant set for $\kappa \geq \aleph_0$ as follows.

Definition 2.2. A set E is κ -invariant, if for any $x, y \in E$ and $S \subseteq T$ with $|S| < \kappa$ there is a point $z \in E$ such that $z_t = x_t$ for every $t \in S$ and $z_t = y_t$ for every $t \in T \setminus S$.

Developing the above-mentioned concepts of Mazur and Hušek, we introduce the following notions.

Definition 2.3. Let \aleph_i and \aleph_j be infinite cardinals, $E \subseteq X$ and $a \in E$. Then E is called

- a) \aleph_i -invariant with respect to the point a , if $x_S^a \in E$ for every $x \in E$ and $S \subseteq T$ with $|S| < \aleph_i$;
- b) (\aleph_i, \aleph_j) -invariant with respect to the point a , if $x_{S_1}^a \in E$ and $x_{T \setminus S_2}^a \in E$ for any point $x \in E$ and for any sets $S_1, S_2 \subseteq T$ with $|S_1| < \aleph_i$ and $|T \setminus S_2| < \aleph_j$.

Obviously, every (\aleph_i, \aleph_j) -invariant set with respect to a is \aleph_i -invariant with respect to a .

Definition 2.4. A topological space X is said to be

- *pseudo- \aleph_1 -compact*, if any locally finite family of open subsets of X is at most countable;
- *hereditarily pseudo- \aleph_1 -compact*, if each subspace of X is pseudo- \aleph_1 -compact.

It is easy to check that continuous mappings preserve the pseudo- \aleph_1 -compactness.

The following theorem gives a characterization of the pseudo- \aleph_1 -compactness of \aleph_0 -invariant sets and is an analogue of the similar result of Noble and Ulmer [11, Corollary 1.5] for products.

Theorem 2.1. Let $(X_t : t \in T)$ be a family of topological spaces, $X = \prod_{t \in T} X_t$, $a \in X$ and let $E \subseteq X$ be an \aleph_0 -invariant set with respect to a . Then the following conditions are equivalent:

- (i) E is pseudo- \aleph_1 -compact;
- (ii) for any finite non-empty set $S \subseteq T$ and for any uncountable family $(U_i : i \in I)$ of open sets U_i in X with $U_i \cap E \neq \emptyset$ the family $(p_S(U_i \cap E) : i \in I)$ is not locally finite in $p_S(E)$.

Proof. (i) \Rightarrow (ii). Let $S \subseteq T$ be a finite non-empty set, $(U_i : i \in I)$ be an uncountable family of basic open sets U_i in X with $U_i \cap E \neq \emptyset$ and let $V_i = p_S(U_i \cap E)$ for each $i \in I$. If the family $(V_i : i \in I)$ is locally finite in $p_S(E)$, then the family $(p_S^{-1}(V_i) \cap E : i \in I)$ is locally finite in E and $U_i \cap E \subseteq p_S^{-1}(V_i) \cap E$ for each $i \in I$, which contradicts to the pseudo- \aleph_1 -compactness of E .

(ii) \Rightarrow (i). Consider an uncountable family $(U_i = \prod_{t \in T} U_i^t : i \in I)$ of basic open sets in X such that $U_i \cap E \neq \emptyset$ for all $i \in I$. By Šanin's lemma [12] we choose a finite set Z and uncountable set $J \subseteq I$ such that $N(U_i) \cap N(U_j) = Z$ for all distinct $i, j \in J$.

Let $V_i = p_Z(U_i \cap E)$ for all $i \in J$. It follows from (ii) that the family $(V_i : i \in J)$ has a cluster point $v \in p_Z(E)$. Take $y \in E$ such that $v = p_Z(y)$ and put $x = y_Z^a$. We shall show that x is a cluster point of $(U_i \cap E : i \in J)$. Indeed, let $W = \prod_{t \in T} W_t$ be a basic open neighborhood of x in X and $V = \prod_{t \in Z} W_t \cap p_Z(E)$.

Choose such an infinite set $K \subseteq J$ that $V \cap V_i \neq \emptyset$ and $N(W) \cap N(U_i) \subseteq Z$ for all $i \in K$. Take an arbitrary $i \in K$ and a point $b \in V \cap V_i$. Consider a point $c \in U_i \cap E$ with $b = p_Z(c)$ and put $d = c_{Z \cup N(U_i)}^a$. Clearly, $d \in U_i$. Since E is \aleph_0 -invariant with respect to a and $c \in E$, $d \in E$. Moreover, $p_Z(d) = p_Z(c) = b \in V$ and $d_t = a_t \in W_t$ for every $t \in N(W) \setminus Z$. Therefore, $d \in W$. Hence, $d \in W \cap E \cap U_i$. \square

The example below shows that the condition (ii) in the previous theorem can not be weakened to the following: *the set $p_S(E)$ is pseudo- \aleph_1 -compact for any non-empty finite set $S \subseteq T$.*

Example 2.1. *There exists an (\aleph_1, \aleph_1) -invariant set $E \subseteq \prod_{t \in T} X_t$ with respect to a point $a \in E$ such that $p_S(E)$ is pseudo- \aleph_1 -compact for any non-empty finite set $S \subseteq T$, but E is not pseudo- \aleph_1 -compact.*

Proof. Let $T = [0, 1]$, $X_0 = \mathbb{P} = \mathbb{R} \times [0, +\infty)$ be the Niemytzki plane [6, p. 21], $X_t = \{0, 1\}$ for each $t \in (0, 1]$, $X = \prod_{t \in T} X_t$ and let $a = (a_t)_{t \in T} \in X$, where $a_t = 0$ for each $t \in (0, 1]$ and $a_0 = (0, 0)$. For each $t \in (0, 1]$ we define $y^{(t)} = (y_s^{(t)})_{s \in T}$ and $z^{(t)} = (z_s^{(t)})_{s \in T} \in X$ as follows:

$$y_s^{(t)} = \begin{cases} 0, & s \in (0, 1] \setminus \{t\} \\ 1, & s = t \\ (t, 0), & s = 0, \end{cases}$$

$$z_s^{(t)} = \begin{cases} 0, & s \in (0, 1] \setminus \{t\} \\ 1, & s = t \\ (0, 0), & s = 0. \end{cases}$$

Consider the (\aleph_1, \aleph_1) -invariant set

$$E = \{y^{(t)} : t \in (0, 1]\} \cup \{z^{(t)} : t \in (0, 1]\} \cup (X_0 \times \prod_{t \in (0, 1]} \{0\})$$

with respect to the point a . Observe that for any finite set $S \subseteq [0, 1]$ the sets $p_S(\{y^{(t)} : t \in (0, 1]\})$ and $p_S(\{z^{(t)} : t \in (0, 1]\})$ are finite and the set $p_S(X_0 \times \prod_{t \in (0, 1]} \{0\})$ is separable. Hence, E satisfies the condition mentioned above. But $(\{y^{(t)}\} : t \in (0, 1])$ is a locally finite family of open sets in E . Therefore, E is not pseudo- \aleph_1 -compact. \square

3. Dependence on countably many coordinates of continuous mappings

Definition 3.1. Let $E \subseteq \prod_{t \in T} X_t$. A function $f : E \rightarrow Y$ depends on a set $S \subseteq T$ [3, p. 231], if for all $x, y \in E$ the equality $p_S(x) = p_S(y)$ implies $f(x) = f(y)$. If $|S| \leq \aleph_0$, then we say that f depends on a countably many coordinates. Similarly, E depends on S , if for all $x \in E$ and $y \in X$ with $p_S(x) = p_S(y)$ we have $y \in E$.

Definition 3.2. A space Y has a regular G_δ -diagonal [14], if there exists a sequence $(G_n)_{n=1}^\infty$ of open subsets of Y^2 such that

$$\{(y, y) : y \in Y\} = \bigcap_{n=1}^\infty G_n = \bigcap_{n=1}^\infty \overline{G_n}. \quad (1)$$

We denote $\sigma(a) = \{x \in X : |t \in T : x_t \neq a_t| < \aleph_0\}$ as in [4].

Theorem 3.1. Let Y be a space with a regular G_δ -diagonal, $(X_t : t \in T)$ be a family of topological spaces, $X = \prod_{t \in T} X_t$, $a \in X$ and let $E \subseteq X$ be a pseudo- \aleph_1 -compact subspace which is (\aleph_1, \aleph_0) -invariant with respect to a . Then for any continuous mapping $f : E \rightarrow Y$ there exist a countable set $T_0 \subseteq T$ and a continuous mapping $f_0 : p_{T_0}(E) \rightarrow Y$ such that $f = f_0 \circ (p_{T_0}|_E)$.

In particular, f depends on countably many coordinates.

Proof. Let $(G_n)_{n=1}^\infty$ be a sequence of open sets in Y^2 which satisfies (1) and let $f : E \rightarrow Y$ be a continuous function. Denote by T_0 the set of all $t \in T$ for which there exist points $x^t, y^t \in E \cap \sigma(a)$ such that

$$x_s^t = y_s^t \text{ for all } s \neq t, \quad (2)$$

$$x_t^t = a_t, \quad (3)$$

$$f(x^t) \neq f(y^t). \quad (4)$$

Assume that T_0 is uncountable and choose an uncountable subset $B \subseteq T_0$ and a number $n_0 \in \mathbb{N}$ such that

$$(f(x^t), f(y^t)) \in Y^2 \setminus \overline{G_{n_0}} \text{ for all } t \in B.$$

Using the continuity of f at x^t and y^t for every $t \in B$, we find such open basic neighborhoods U^t and V^t of x^t and y^t , respectively, that

$$p_s(U^t) = p_s(V^t) \text{ for } s \neq t, \quad (5)$$

$$f(U^t \cap E) \times f(V^t \cap E) \subseteq Y^2 \setminus \overline{G_{n_0}}. \quad (6)$$

Since E is pseudo- \aleph_1 -compact and the family $(V^t \cap E : t \in B)$ is uncountable, there exists a point $x^* \in E$ such that for any basic open neighborhood W of x^* the set $C_W = \{t \in B : V^t \cap E \cap W \neq \emptyset\}$ is infinite. The continuity of f at x^* implies that there is such a basic neighborhood W of x^* that $f(W \cap E) \times f(W \cap E) \subseteq G_{n_0}$. Notice that $C = C_W \setminus N(W) \neq \emptyset$. Fix $t \in C$ and $y \in V^t \cap E \cap W$. Let $x = y_{T \setminus \{t\}}^a$. Then (3) and (5) imply that $x \in U^t$. Since E is (\aleph_1, \aleph_0) -invariant with respect to a , $x \in E$. Moreover, $x \in W$, since $t \notin N(W)$. Then $(f(x), f(y)) \in G_{n_0}$, which contradicts (6). Hence, the set T_0 is countable.

We show that f depends on T_0 . To do this it is sufficient to check the equality $f(x) = f(x_{T_0}^a)$ for every $x \in E$. Consider the case $x \in E \cap \sigma(a)$. Let $\{t \in T \setminus T_0 : x_t \neq a_t\} = \{t_1, \dots, t_m\}$. Then

$$\begin{aligned} f(x) &= f(x_{T \setminus \{t_1\}}^a) = f((x_{T \setminus \{t_1\}}^a)_{T \setminus \{t_2\}}^a) = \dots = \\ &= f(((x_{T \setminus \{t_1\}}^a) \dots)_{T \setminus \{t_m\}}^a) = f(x_{T_0}^a). \end{aligned}$$

Now let $x \in E$. Notice that $E \cap \sigma(a)$ is a dense set in E . Indeed, if $b = (b_t)_{t \in T} \in E$ and W is a basic open neighborhood of b in X , then $b_{N(W)}^a \in W \cap E \cap \sigma(a)$. Hence, there exists a net (x_i) of points $x_i \in E \cap \sigma(a)$ such that $\lim_i x_i = x$. Then $\lim_i (x_i)_{T_0}^a = x_{T_0}^a$. It follows from the continuity of f that

$$f(x) = f(\lim_i x_i) = \lim_i f(x_i) = \lim_i f((x_i)_{T_0}^a) = f(\lim_i (x_i)_{T_0}^a) = f(x_{T_0}^a).$$

Consider the function $f_0 : p_{T_0}(E) \rightarrow Y$ defined by $f_0(z) = f(x)$, if $z = p_{T_0}(x)$ for $x \in E$. Observe that f_0 is defined correctly, because f depends on T_0 . It remains to prove that f_0 is continuous on $p_{T_0}(E)$. Fix $z \in p_{T_0}(E)$ and a net (z_i) of points $z_i \in p_{T_0}(E)$ such that $\lim_i z_i = z$. Take $x \in E$ and $x_i \in E$ with $z = p_{T_0}(x)$ and $z_i = p_{T_0}(x_i)$. Let $y_i = (x_i)_{T_0}^a$ and $y = x_{T_0}^a$. Then $y_i, y \in E$ and $\lim_i y_i = y$. Moreover, since f is continuous at y , we have

$$\lim_i f_0(z_i) = \lim_i f(x_i) = \lim_i f(y_i) = f(y) = f(x) = f_0(z).$$

Hence, f_0 is continuous at z . \square

Notice that the proof of the dependence of f on T_0 in Theorem 3.1 is similar to the proof of Lemma 2.32 and Lemma 2.27(a) in [1].

Theorem 3.2. *Let $(X_t : t \in T)$ be an uncountable family of topological spaces, $X = \prod_{t \in T} X_t$, $a \in X$ and let $E \subseteq X$ be an (\aleph_1, \aleph_0) -invariant set with respect to a . Consider the following conditions:*

- (i) E is pseudo- \aleph_1 -compact;
- (ii) for any space Y with a regular G_δ -diagonal and for any continuous mapping $f : E \rightarrow Y$ there exist a countable set $T_0 \subseteq T$ and a continuous mapping $f_0 : p_{T_0}(E) \rightarrow Y$ such that $f = f_0 \circ (p_{T_0}|_E)$;
- (iii) for any continuous function $f : E \rightarrow \mathbb{R}$ there exist a countable set $T_0 \subseteq T$ and a continuous mapping $f_0 : p_{T_0}(E) \rightarrow \mathbb{R}$ such that $f = f_0 \circ (p_{T_0}|_E)$.

Then (i) \Rightarrow (ii) \Rightarrow (iii).

If E is completely regular and

- (iv) for any non-empty open set U in E there exists an uncountable set $T_U \subseteq T$ such that for every $t \in T_U$ there are $y^{(t)} = (y_s^{(t)})_{s \in T}, z^{(t)} = (z_s^{(t)})_{s \in T} \in U$ with $y_t^{(t)} \neq z_t^{(t)}$ and $y_s^{(t)} = z_s^{(t)}$ for every $s \in T \setminus \{t\}$,

then (iii) \Rightarrow (i).

Proof. The implication (i) \Rightarrow (ii) follows from Theorem 3.1.

The implication (ii) \Rightarrow (iii) is obvious.

Prove that (iii) \Rightarrow (i). Suppose that E is not pseudo- \aleph_1 -compact and choose a locally finite in E family $(U_\alpha : \alpha < \omega_1)$ of non-empty open sets U_α . Note that U_α may be taken to be disjoint. Indeed, let $(V_i : i \in I)$ be a locally finite family of non-empty open subsets of E with $|I| > \aleph_0$. For every $i \in I$ we choose a non-empty open set $W_i \subseteq V_i$ and a finite set $J_i \subseteq I$ such that $W_i \subseteq \bigcap_{j \in J_i} V_j$ and $W_i \cap V_j = \emptyset$ for all $j \in I \setminus J_i$. Since $i \in J_i$ for every $i \in I$, $\bigcup_{i \in I} J_i = I$.

Now we take a uncountable set $I_0 \subseteq I$ such that all the sets J_i from the family $(J_i : i \in I_0)$ are different. Then the uncountable family $(W_i : i \in I_0)$ consists of mutually disjoint elements.

Since E is completely regular, we may assume that all the sets U_α are functionally open. For every $\alpha < \omega_1$ take a continuous function $f_\alpha : E \rightarrow [0, 1]$ such that $U_\alpha = f_\alpha^{-1}((0, 1])$. Since T_{U_α} is uncountable, we may construct a family $(t_\alpha : \alpha < \omega_1)$ of distinct points $t_\alpha \in T_{U_\alpha}$. According to (iv) we choose for every $\alpha < \omega_1$ points $y^{(\alpha)} = (y_s^{(\alpha)})_{s \in T}, z^{(\alpha)} = (z_s^{(\alpha)})_{s \in T} \in U_\alpha$ such that $y_{t_\alpha}^{(\alpha)} \neq z_{t_\alpha}^{(\alpha)}$ and $y_s^{(\alpha)} = z_s^{(\alpha)}$ for every $s \in T \setminus \{t_\alpha\}$. Now for every $\alpha < \omega_1$ we choose a continuous function $g_\alpha : E \rightarrow [0, 1]$ such that $g_\alpha(y^{(\alpha)}) = 1$ and $g_\alpha(z^{(\alpha)}) = 0$.

Consider the continuous function $f : E \rightarrow [0, 1]$, $f(x) = \sum_{\alpha < \omega_1} f_\alpha(x)g_\alpha(x)$.

Since the sets U_α are mutually disjoint,

$$f(y^{(\alpha)}) - f(z^{(\alpha)}) = f_\alpha(y^{(\alpha)})g_\alpha(y^{(\alpha)}) - f_\alpha(z^{(\alpha)})g_\alpha(z^{(\alpha)}) = f_\alpha(y^{(\alpha)}) > 0.$$

Hence, $f(y^{(\alpha)}) \neq f(z^{(\alpha)})$ for every $\alpha < \omega_1$. Since the set $\{t_\alpha : \alpha < \omega_1\}$ is uncountable, the function f does not satisfy (iii). \square

4. Functionally measurable sets

Proposition 4.1. *Let E be a subset of a topological space $X = \prod_{t \in T} X_t$ such that for any continuous function $f : E \rightarrow \mathbb{R}$ there exist a countable set $T_0 \subseteq T$ and a*

continuous mapping $f_0 : p_{T_0}(E) \rightarrow \mathbb{R}$ with $f = f_0 \circ (p_{T_0}|_E)$ and let $0 \leq \alpha < \omega_1$. Then for any set A of the functionally additive (multiplicative) class α in E there exists a countable set $T_0 \subseteq T$ such that A depends on T_0 and $p_{T_0}(A)$ is of the functionally additive (multiplicative) class α in $p_{T_0}(E)$.

Proof. Let $\alpha = 0$. We consider the case when a set A is functionally open in E . Then $A = f^{-1}((0, +\infty))$ for some continuous function $f : E \rightarrow \mathbb{R}$. Take a countable set $T_0 \subseteq T$ and a continuous mapping $f_0 : p_{T_0}(E) \rightarrow \mathbb{R}$ with $f = f_0 \circ (p_{T_0}|_E)$. Then the set $p_{T_0}(A) = f_0^{-1}((0, +\infty))$ is functionally open in $p_{T_0}(E)$. Moreover, if $x \in A$ and $y \in E$ with $p_{T_0}(x) = p_{T_0}(y)$, then $f(y) = f(x) > 0$. Therefore, $y \in A$, which implies that A depends on T_0 .

Assume that the proposition is true for all $\alpha < \beta$ and consider a set A of the functionally additive class α in E . Then $A = \bigcup_{n=1}^{\infty} A_n$, where A_n is of the functionally multiplicative class $\alpha_n < \alpha$ for every n . By the assumption for every n there exists a countable set $T_n \subseteq T$ such that A_n depends on T_n and $p_{T_n}(A_n)$ belongs to the functionally multiplicative class α_n in $p_{T_n}(E)$. Notice that the set $p_{T_0}(A_n)$ is of the functionally multiplicative class α_n in $p_{T_0}(E)$ for every n . Then $p_{T_0}(A) = \bigcup_{n=1}^{\infty} p_{T_0}(A_n)$ is of the functionally additive class α in $p_{T_0}(E)$. \square

Definition 4.1. Let $0 \leq \alpha < \omega_1$. A space X is α -universal, if any subset of X is α -embedded in X .

Clearly, every perfectly normal space is α -universal for any $\alpha < \omega_1$.

Proposition 4.2. Let $0 \leq \alpha < \omega_1$, $(X_t)_{t \in T}$ be a family of topological spaces such that every countable subproduct is α -universal, $X = \prod_{t \in T} X_t$ and let $E \subseteq X$ be such a set as in Proposition 4.1. Then E is an α -embedded set in X .

Proof. Let $A \subseteq E$ be a set of the functionally multiplicative class α in E . According to Proposition 4.1 there exist a countable set $T_0 \subseteq T$ such that A depends on T_0 and $A_0 = p_{T_0}(A)$ is of the functionally multiplicative class α in $E_0 = p_{T_0}(E)$. Since $X_0 = \prod_{t \in T_0} X_t$ is α -universal, the set E_0 is α -embedded in X_0 . Hence, there exists a set B_0 of the functionally multiplicative class α in X_0 such that $B_0 \cap E_0 = A_0$. Let $B = p_{T_0}^{-1}(B_0)$. Then B is of the functionally multiplicative class α in X , because the mapping p_{T_0} is continuous. Moreover, it is easy to see that $B \cap E = A$. \square

Proposition 4.3. Let $0 \leq \alpha < \omega_1$, $X = \prod_{t \in T} X_t$ be a pseudo- \aleph_1 -compact space, where $(X_t)_{t \in T}$ is a family of spaces such that every countable subproduct is α -universal and hereditarily pseudo- \aleph_1 -compact. Then any functionally measurable set $E \subseteq X$ is α -embedded in X .

Proof. Consider a functionally measurable set $E \subseteq X$. Without loss of generality, we may assume that E belongs to the functionally multiplicative class β for

some $0 \leq \beta < \omega_1$. Take a function $f \in B_\beta(X)$ such that $E = f^{-1}(0)$. Since X is pseudo- \aleph_1 -compact, Theorem 2.3 from [11] implies that there exists a countable set $T_0 \subseteq T$ such that for all $x \in E$ and $y \in X$ the equality $p_{T_0}(x) = p_{T_0}(y)$ implies that $y \in E$. Let $E_0 = p_{T_0}(E)$. Then

$$E = E_0 \times \prod_{t \in T \setminus T_0} X_t.$$

Since $\prod_{t \in T_0 \cup S} X_t$ is a hereditarily pseudo- \aleph_1 -compact space, $E_0 \times \prod_{t \in S} X_t$ is pseudo- \aleph_1 -compact space for any finite set $S \subseteq T \setminus T_0$. Hence, by [11, Corollary 1.5] the set E is pseudo- \aleph_1 -compact. Therefore, E satisfy the condition of Proposition 4.1 by Theorem 3.1 applied to the whole product $E_0 \times \prod_{t \in T \setminus T_0} X_t$.

It remains to use Proposition 4.2. \square

The following result implies a positive answer to Question 8.1 from [8].

Corollary 4.1. *Let $(X_t)_{t \in T}$ be a family of separable metrizable spaces. Then every functionally measurable subset of $X = \prod_{t \in T} X_t$ is α -embedded in X for any $0 \leq \alpha < \omega_1$.*

Proof. It follows from Proposition 4.3 and the fact that any countable product of separable metrizable spaces is separable and metrizable, consequently, α -universal and hereditarily pseudo- \aleph_1 -compact. \square

5. The construction of α -embedded sets

Theorem 5.1. *For every $0 \leq \alpha < \omega_1$ there exist a completely regular space X with an $(\alpha + 1)$ -embedded subspace $E \subseteq X$ which is not α -embedded.*

Proof. Fix $\alpha < \omega_1$. Let $X_0 = [0, 1]$, $X_t = \mathbb{N}$ for every $t \in (0, 1]$, $Y = \prod_{t \in (0, 1]} X_t$ and $X = [0, 1] \times Y = \prod_{t \in [0, 1]} X_t$.

According to [9, p. 371] there exists a set $A_1 \subseteq [0, 1]$ of the additive class α which does not belong to the multiplicative class α . Let $A_2 = [0, 1] \setminus A_1$.

For $i = 1, 2$ put

$$F_i = \bigcap_{n \neq i} \{y = (y_t)_{t \in (0, 1]} \in Y : |\{t \in (0, 1] : y_t = n\}| \leq 1\}.$$

It is easy to see that F_1 and F_2 are closed disjoint subsets of Y . Let $B_i = A_i \times F_i$ for $i = 1, 2$ and $E = B_1 \cup B_2$. Then B_1 and B_2 are disjoint closed subsets of E .

Claim 1. The set B_i is α -embedded in X for every $i = 1, 2$.

Proof. We show that B_1 is pseudo- \aleph_1 -compact (for the set B_2 we argue completely similarly). Since A_1 is separable, it is enough to check that F_1 is pseudo- \aleph_1 -compact. Notice that the set F_1 is (\aleph_1, \aleph_1) -invariant with respect to

the point $a = (a_t)_{t \in (0,1]}$, where $a_t = 1$ for every $t \in (0,1]$. Since for any finite set $S \subseteq (0,1]$ the space $\prod_{t \in S} X_t$ is countable, the set F_1 satisfies condition (ii) of

Theorem 2.1. Then by Theorem 2.1 the set F_1 is pseudo- \aleph_1 -compact.

Now observe that each set B_i is (\aleph_1, \aleph_1) -invariant with respect to the point $a^i = (a_t^i)_{t \in [0,1]}$, where $a_t^i = i$ for all $t \in (0,1]$ and $a_0^i \in A_i$. It remains to apply Theorem 3.1 and Proposition 4.2.

Claim 2. The set E is not α -embedded in X .

Proof. Assume the contrary and choose a set H of the functionally multiplicative class α in X such that $H \cap E = B_1$. It follows from Proposition 4.1 that there is a countable set $S = \{0\} \cup T$, where $T \subseteq (0,1]$, such that H depends on S . Let $y_0 \in Y$ be such a point that $p_T(y_0)$ is a sequence of distinct natural numbers which are not equal to 1 or 2. Take $y_1 \in F_1$ and $y_2 \in F_2$ with $p_T(y_0) = p_T(y_1) = p_T(y_2)$. Then for all $x \in A_1$ we have $(x, y_1) \in H$ and, consequently, $(x, y_0) \in H$. Moreover, for all $x \in A_2$ we have $(x, y_2) \notin H$ and, consequently, $(x, y_0) \notin H$. Hence, $A_1 \times \{y_0\} = ([0,1] \times \{y_0\}) \cap H$. Therefore, $A_1 \times \{y_0\}$ is of the functionally multiplicative class α in X , which implies that the set A_1 belongs to the functionally multiplicative class α in $[0,1]$, a contradiction.

Claim 3. The set E is $(\alpha + 1)$ -embedded in X .

Proof. Let C be a set of the functionally multiplicative class $(\alpha + 1)$ in E . Denote $E_i = A_i \times Y$ for $i = 1, 2$. Then E_1 is of the functionally additive class α and E_2 is of the functionally multiplicative class α in X . For $i = 1, 2$ put $C_i = C \cap B_i$. Since each of the sets C_i is of the functionally multiplicative class $(\alpha + 1)$ in the α -embedded set B_i in X , there exists a set D_i of the functionally multiplicative class $(\alpha + 1)$ in X such that $D_i \cap B_i = C_i$. Let $D = (D_1 \cap E_1) \cup (D_2 \cap E_2)$. Then D is a set of the functionally multiplicative class $(\alpha + 1)$ in X and $D \cap E = C$. \square

Notice that the sets F_i were first defined by A. Stone [13] in his proof of non-normality of the uncountable power \mathbb{N}^τ of the space \mathbb{N} of natural numbers.

Acknowledgement

The authors express gratitude to the referees for careful reading and comments that helped to improve the paper.

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